Strategies to

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An important question for mathematics teachers is this: "How can we help students learn mathematics to solve everyday problems, rather than teaching them only to memorize rules and practice mathematical procedures?" Teaching students using modeling activities can help them learn mathematics in real-world problem-solving situations that are useful outside school (Lesh and Doerr 2003b).

Lesh and Lehrer (2003) defined mathematical modeling as a process of designing and revising representations to solve problems. Modeling, also described in the Common Core State Standards for Mathematics (CCSSM), is one of the eight Standards for Mathematical Practice: "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (CCSSI 2010, p. 7). By including

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Students' Mathematical Modeling

To witness a spike in students' mathematical abilities, explore a model-eliciting activity.

modeling as a mathematical practice, CCSSM invites teachers to strategize ways to develop students' mathematical modeling.

Because CCSSM provides a brief description of modeling, teachers might interpret modeling in various ways (e.g., the demonstration of a lesson, construction of a physical object, or simulating a process) (Imm and Lorber 2013). Collaboration among teachers and researchers to implement modeling activities can be a valuable way to promote a better understanding of modeling for teachers. Through collaboration, researchers can bring relatively abstract and isolated knowledge to the table in a way that makes these ideas more applicable to the classroom (Hiebert, Gallimore, and Stigler 2002). At the same time, teachers can be involved in making decisions about what change is relevant for their students' learning (Richardson 1990). This article describes the strategies that two middle school teachers and their students used while working on modeling activities with me, a researcher, over a semester.

DESCRIBING OUR JOURNEY

Two eighth-grade teachers, Kate and Megan (both names are pseudonyms), worked with me when I volunteered to serve as a visiting scholar to help

In the place of word-heavy activity sheets, PowerPoint was used to display images and other information for special needs students.

integrate research and new approaches into mathematics classrooms at a local, Midwestern school with a diverse student population (i.e., about 60 percent Caucasian, 12 percent African American, and 20 percent Hispanic). Approximately 70 percent of the students participated in the free or reduced-price federal lunch program. Both teachers taught five different classes for seven periods every day: two inclusion classes, two general education classes, and one co-taught inclusion class. Inclusion classes contained low-achieving students and students with special needs, whereas general education classes had average-achieving students. Other teachers at the middle school taught the advanced classes with high-achieving students. The two teachers and I decided to co-develop and co-teach modeling lessons using Model-Eliciting Activities (MEAs) that we implemented three times during my eleven weeks at the school.

MODEL-ELICITING ACTIVITIES (MEAs)

An MEA is a mathematical task wherein teams of three to five students solve realistic mathematical problems (Lesh et al. 2000). When students engage with an MEA, they describe, evaluate, and revise models over one or two class periods. For example, in the Volleyball MEA (Lesh and Doerr 2003a), teams of students are asked to write a letter describing a process for creating equal teams that would accommodate a very large number of volleyball players. **Figure 1** shows a modified version of the Volleyball MEA that Kate and Megan taught in their classrooms. Lesh and Doerr (2003a) emphasize that MEAs are designed to help students develop a sharable, modifiable, and reusable tool, rather than answer narrowly specified story problems. Their overall design principles for an MEA are summarized in **table 1** (Lesh et al. 2000).

MODIFYING THE MEA

When the original Volleyball MEA was piloted with other students, it took more than 50 minutes, which was one class period for both teachers. We discussed to what extent the lesson, as designed, would be appropriate for the students, especially those with special needs. Kate and Megan were concerned that their students might be overwhelmed by all the data presented in the original MEA. Because reducing the amount of data would not violate the design principles for an MEA, we changed the number of players from eighteen to eight and the number of spike-result attempts from five to three, as shown in **figure 1**. Providing activity sheets containing a lot of words might challenge students, so we decided to use PowerPoint® to show both images and a condensed version of the information.

USING VARIOUS STRATEGIES

When we modified, implemented, and reflected on the Volleyball activity, we used strategies that supported students' development of the modeling process. The strategies presented in this article were each identified to help students meet the goals of the design principles shown in **table 1**. The following examples of these strategies resulted from an analysis of audiotaped interviews and discussions with the two teachers and their students' written work.

1. Reality: Ask students questions so they understand the task on the basis of their own real-life experiences.

Table 1 Teachers can consider these principles when they design or modify a Model-Eliciting Activity, or MEA.

Principle	Description
Reality	Students understand the realistic problem on the basis of extensions of their previous experiences.
Effective prototype	Students provide a mathematically significant and simple model for the complex situations.
Model generalizability	Students develop knowledge that can be used in other similar contexts.
Model documentation	Students produce documentation of their thinking to express possible solution paths to meet the goals.
Model construction	Students construct a model as they describe, extend, verify, or revise the model.
Self-evaluation	Students evaluate their answers on the basis of the statement of the problem, including the criteria, and go beyond their initial thoughts.

Fig. 1 The Volleyball MEA asked students to mathematically interpret and describe a complex real-life situation.

For years, volleyball has been the focal point of the fall semester at several middle schools. During a summer volleyball camp, coaches help each team continue their winning ways. One of the students who participated in the volleyball camp last year recalled losing one match 15-2. Because of the lack of competition, interest in the camp is decreasing. The organizers of the volleyball camp wish to offer more competition. They have compiled information about some of the players from tryouts and from the coaches.

Data from Volleyball Tryouts

Name	Height of Player	Vertical Jump in Inches	40 Meter Dash in Seconds	Spike Results (Out of 3 Attempts)
Gertrude	6′1″	20	6.21	Returned / kill / kill
Beth	5′2″	25	5.98	Kill / returned / out of bounds
Jill	5′10″	24	6.44	Out of bounds / returned / returned
Amy	5′10″	27	6.01	Kill / kill / in the net
Ana	5′6″	25	6.95	Out of bounds / in the net / returned
Kate	5′8″	17	7.12	Kill / returned / kill
Rhonda	5′3″	21	6.34	Out of bounds / kill / in the net
Christina	5′5″	23	7.34	In the net / kill / kill

Spike Results Vocabulary

Kill: The other team was unable to return the ball.
Out of bounds: The hitter spiked the ball out of bounds.
Returned: The other team returned the spike.
In the net: The hitter failed to hit the ball over the net.

The camp organizers need you to split the players into two equal teams. In addition to forming these two teams, they need you to write a letter to them, describing how you created your two equal teams. They will use your process for the next camp when they need to split a large number of players into equal teams.

2. Effective prototype: Guide students through questioning to develop a mathematically significant model.
3. Model generalizability: Remind students to construct a model that is generalizable.
4. Model documentation: Require students to document their process.
5. Model construction: Facilitate discussion in which students present the process of developing their model.
6. Self-evaluation: Encourage students to use peer-review forms to evaluate their own responses.

These teacher strategies are discussed in greater detail in the following sections.

Reality

To help students engage with the problem context, we showed them a short video clip at the beginning of the lesson and asked them if they had ever played volleyball. Students were also introduced to the problem shown in the PowerPoint slides and asked the following questions to help them make sense of the real-life situation:

- “What problem is the camp having?” and
- “Why do you think the camp had a lack of competition last year?”

We then asked if they knew volleyball terminology, such as *kill, out of bounds, returned,* and *in the net*. Rather than having the teacher explain them, we asked some of the volunteering students to explain the meanings of the terms to the whole class.

Effective Prototype

When students read the table in **figure 1**, we wanted them to mathematically consider the real-life context. Students were asked:

- Who is the tallest player among those listed in the table?
- Which player can jump the highest?
- Is this the same person as the player who can reach the highest point? Why or why not?

After teaching the lesson, Megan described another example of how

she encouraged students to develop a mathematically significant model:

> With my fifth hour, we threw in some more concepts like "Eight out of ten serves equals what percent?" Eighty percent. And we kind of went into, "What's this group's percentage for spike results? What's this team's percentage on serve results?"

These questions helped students use the mathematical concepts that underlie the task to create models for complex situations. **Figure 2**, for example, shows that a student recorded spike results (e.g., 2/5, 4/5) and used them to divide the players into two teams.

Fig. 2 A student used fractions to express the serve results and spike results of players.

Name	Height of Player	Vertical Jump in inches	40 meter dash in seconds	Serve Results (number of serves that were successful out of 10 attempts)	Spike Results (out of 5 attempts)
Gertrude	6'1"	20	6.21	8/10	Returned / Kill / Kill / In the net / Returned 2/5
Beth	5'2"	25	5.98	7/10	Kill / Returned / Out of Bounds / Kill / In the Net 2/5
Jill	5'10"	24	6.44	8/10	Out of Bounds / Returned / Returned / Kill / In the Net 2/5
Amy	5'10"	27	6.01	9/10	Kill / Kill / In the Net / Kill / Returned 3/5
Ana	5'6"	25	6.95	10/10	Out of Bounds / In the Net / Returned / Returned / Kill 1/5
Kate	5'8"	17	7.12	6/10	Kill / Returned / Kill / Returned / Kill 3/5
Rhonda	5'3"	21	6.34	5/10	Out of Bounds / Kill / In the Net / In the Net / Returned 1/5
Christina	5'5"	23	7.34	8/10	In the Net / Kill / Kill / Kill / Kill 4/5

Use this space for organizing your process

team #1
93" Gertrude 2/5
87" Beth 2/5
85" Kate 3/5
91" Ana 1/5

team #2
Jill 2/5 94"
Amy 3/5 97"
Rhonda 1/5 84"
Christina 4/5 88"

Model Generalizability

Even though the problem states that "They will use your process for the next camp when they need to split a large number of players into equal teams," students often focused on creating their two equal teams, rather than developing a model that could be used to split a large number of players. Kate explained how she reminded students that their process should work for twenty or more players:

> A lot of them think of what's in front of them. They don't think of the aftereffect or what might have to come next. A lot of them would think here and now. So I keep reminding them, so then once they get focused in one thing, they would focus on just eight kids and they wouldn't think about "Why am I putting these kids into those groups?" So I remind them this might be twenty or a hundred, that way they would go back to the concrete reasoning of why I'm putting these kids in these groups. And maybe ask them to write out more than just the kids' names.

Kate helped students move beyond what was immediately in front of them to think about the broader applicability of their solutions. Her strategy helped students develop a generalizable model that works for other similar situations (see **fig. 3**).

Fig. 3 One team's mathematical interpretation of the problem context was explored by listing the strategies used.

Strategies
- Multiplied height of player by vertical jump in inches to see who can reach highest point
- Numbered students in order by how fast they can run
- Numbered in order by spike & serve results
- Divided players by worst and best
- Made groups by pairing worst and best players

Model Documentation

During the group work, students were asked to document all the processes they used. They added this information under the table in a section labeled "Use this space for organizing your process" (see **fig. 2**). Documenting their work reminded students of their thought processes when discussing the work in their groups and with the whole class. Each group also wrote a letter to the camp organizer, which included the students' step-by-step strategy to split the teams (see **fig. 3**). This documentation helped students summarize their work and explain their mathematical and nonmathematical interpretation of the problem context.

Model Construction

During this activity, students had several opportunities to provide an explicit explanation for a mathematically significant context. Megan described

how she encouraged students to discuss in their own groups.

> Because what we discussed earlier with the groups, a lot of them just sat there, they didn't do anything. So if you make them think individually, then they have something to share, then they have something to discuss, whereas if you just say, "Here's the information, discuss it," two kids talk, the other kids don't do anything. So if they have something already written in front of them, they have something to share. . . . When it came to the discussion time, they were like, "Why did you pick those four?" "How did you pick those four?" They had a really good discussion. And . . . they were arguing who's right, who's wrong. It was neat.

After the discussions, students presented their work. We selected three or four teams that had created different models to solve the problem so that the other students could see that multiple reasonable solutions were possible.

Self-Evaluation

After the selected teams presented their work, the other students wrote on their reflection forms, including what they thought would be the best way for the camp organizer to fairly divide the campers into two teams (see **fig. 4**). This process helped the listeners reflect on their answers by comparing the presenters' solutions with their own solutions, which then informed their revisions. For example, one student realized the possibility of considering all the data after seeing the other presentations (see **fig. 4**). By using this form, teachers gave students the opportunity to assess their solutions and then revise their own model if desired.

RELATING STRATEGIES TO CCSSM

Students presented their mathematical strategies to both their small group and the whole class while engaging with the modeling activity. Their strategies revealed that they performed several of the mathematical practices described in CCSSM. First, many of the students seemed to "make sense of problems and persevere in solving them" (CCSSI 2010, p. 6). For example, students were introduced to the problem contexts and analyzed the given stories or data, constraints, and goals. When students were stuck on a problem, they were encouraged to think about the problem again before asking questions of the teacher. To present and reflect on their procedures, they had to make sense of their own approaches and also understand the approaches used by their classmates. Such comparisons helped them better understand the problem and suggested an integrated approach to solve the problem when they reflected on the activity.

Second, the MEA helped the students "construct viable arguments and critique the reasoning of others" (CCSSI 2010, pp. 6-7). Students communicated their solutions with others, asked questions during the presentations, and justified their reasons. For example, once a group presented its work, teachers asked the other students, "What questions do you have?" By encouraging students to direct their questions to the presenters, teachers allowed the students to make sense of others' work when the presenters justified their procedures and solutions. This process gave students the opportunity to compare the effectiveness of others' solutions with their own and make modifications if necessary. Furthermore, students described and justified their mathematical procedures in writing by composing a letter to camp organizers.

The students were also encouraged to "attend to precision" (CCSSI 2010, p. 7). Students had to be careful about units of measure (e.g., feet, inches) and needed to express numerical answers that were appropriate for the problem contexts. For example, **figure 2** shows how a student changed feet to inches to combine the height of a player and the vertical jump in inches. When the student explained her solution to her group members, she had to communicate precisely with others so that they

Fig. 4 Students thought about their own answers when they filled out the reflection form.

1. After seeing the other presentations, what do you think would be the best way for your client (the organizers of the volleyball summer camp) to fairly divide the campers into two teams?

By consider all of the data and split the even players into different teams.

2. After solving this activity, circle the score that best describes how well you understand the mathematical ideas you used.

Not at all A little bit Some Most of it All of it

Explain why you feel that way:

I didn't have any trouble trying to figure out how someone got something.

could produce a single group letter integrating their approaches.

DEVELOPING MODELING PRACTICE

Over the semester, teachers interpreted student work and developed assessment tasks that they used in their own practice. They addressed variability in the classroom, such as working with diverse students and classroom environments. Teachers also evaluated their own instructional goals and assessment strategies to plan subsequent lessons. This process of observing, developing, evaluating, and modifying classroom activities is considered a model-eliciting activity at the teacher level (Doerr and Lesh 2003; Lesh and Kelly 2000). As teachers went through this process, students also expressed, tested, and revised their conceptual models as they constructed solutions to real-world problems. When teachers set out to engage students in this modeling practice, students also ended up being engaged in other practices, such as persevering in solving problems, communicating mathematically, and attending to precision. This multitiered, experimental teaching model (Lesh and Kelly 2000) has the potential to develop students' modeling skills as teachers develop ways to teach modeling in their classrooms.

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